

**MEASUREMENT OF VIBRATION WAVEFORMS
USING MULTIPLEX HOLOGRAPHY**

von

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aus

Newton, Mass. / U. S. A.

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von
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CONTENTS

	Page
I Introduction	3
II A Review of Holographic Vibration Analysis Methods	5
III Theoretical Development of the Multiplex Approach	14
A) The Object Motion - Object Wave Transformation	14
B) The Reverse Transformation	16
C) The Holographic Measurement Process	18
D) Non-Harmonic Vibration Analysis	20
IV Experimental Verification	24
A) Intensity Measurements of the Coefficients, $ U_n ^2$	24
1. The Experimental Setup	24
2. The Results	26
B) Phase Measurements and Vibration Waveform Reconstruction	29
1. The Experimental Setup	30
2. Hologram Recording	35
3. The Phase Measurement Process	36
4. The Results	37
V Conclusion	42
Appendix 1: A Probabilistic Method for Calculation of the Coefficients U_n	44
Appendix 2: Expansion of the Coefficients U_n in Terms of Bessel Functions	46
Appendix 3: Optical Display of Object Motion Parameters	48
Appendix 4: The Small Vibration Requirement	51
Appendix 5: Calculation of Phase from Intensity Measurements	54
References	56

ZUSAMMENFASSUNG

Das Ziel dieser Arbeit ist es, mechanische Schwingungen mit Amplituden von etwa 10^{-4} mm kontaktlos zu vermessen. Die Schwingungsform ist nicht auf den Verlauf des Sinus beschränkt. Die entwickelte Methode basiert auf der Holographie.

Ein Hologramm ist bekanntlich die photographische Aufzeichnung von Interferenzstreifen. Wenn während der photographischen Belichtung eine der optischen Komponenten schwingt mit einer Amplitude von nur einem Bruchteil der Wellenlänge des Lichtes, dann verringert sich der Kontrast der Interferenzstreifen. Meistens ist das unerwünscht. Aber Powell und Stetson (Literaturangaben im Hauptteil) haben die Kontrastverminderung benutzt, um Schwingungen zu untersuchen, z. B. an Turbinen und Gitarren.

Bei Powell und Stetson durfte die Schwingung nur sinusförmig sein. Wir dagegen wollen auch nicht-sinusförmige Schwingungen vermessen können, bei denen also mehr Parameter (höhere Fourier-Koeffizienten) unbekannt sind:

$$\Phi(x,t) = \sum_{-\infty}^{\infty} A_n(x) e^{in\omega t}$$

Daher genügt jetzt nicht mehr ein einzelnes Hologramm, sondern mehrere Hologramme müssen gleichzeitig als "Multiplex-Hologramm" aufgenommen werden. Das Problem verkompliziert sich weiter dadurch, daß die optisch-holographische Methode nicht direkt auf den mechanischen Schwingungsverlauf $\Phi(x,t)$ anspricht, sondern auf die komplexe Lichtamplitude:

$$U(x,t) = e^{i\Phi(x,t)}$$

Genauer gesagt: unsere Methode liefert die Fourier-Koeffizienten U_m , Ψ_m der zeitlichen Fourier-Reihe der Lichtamplitude:

$$U(x,t) = \sum_{-\infty}^{\infty} |U_m(x)| e^{i[m\omega t + \Psi_m(x)]}$$

Bei der Auswertung muß also noch von den "optischen Fourier-Koeffizienten" U_m , Ψ_m auf die "mechanischen Fourier-Koeffizienten" geschlossen werden.

Unsere Methode könnte man "Multiplex-Holographie mit synchronisierten Frequenzverschiebungen" nennen. Während unsere Vorgänger Powell und Stetson nur

$$\int U(x,t) dt = \int e^{i\Phi(x,t)} dt$$

erhalten, bekommen wir gleichzeitig für mehrere m-Werte:

$$\int U(x,t) e^{-im\omega t} dt = |U_m(x)| e^{i\Psi_m(x)}$$

Die wesentliche Komponente in unserem holographischen System ist ein bewegtes Gitter, das zwei Aufgaben erfüllt. Einmal wirkt es als "Mehrfach-Strahlteiler", indem es durch Beugung gleichzeitig mehrere Hologramme vom gleichen Objekt erzeugt. Andererseits bewirkt die Bewegung des Gitters, daß das Hologramm der m-ten Gitterbeugungsordnung eine Zeitfrequenzverschiebung (Doppler-Effekt) von $-m\omega$ enthält, wie es in der zuletzt erschienenen Gleichung formuliert wurde.

Das Experiment erforderte große Sorgfalt, weil selbst kleinste Erschütterungen, die Turbulenz der Luft und "drifts" der verschiedenen "power supplies" bei Messungen von sehr kleinen Schwingungsamplituden (z. B. $\lambda/20$) kritisch sind. Deshalb wurde das Experiment im wesentlichen ferngesteuert und von einem kleinen digitalen Computer überwacht.

I INTRODUCTION

In the present volume we are concerned with the measurement of the vibration waveform of an object (such as a loudspeaker, turbine blade, or guitar body) in an arbitrary (generally non-sinusoidal) state of vibration. In the last ten years, holographic methods have been developed capable of accurately determining the amplitude of vibration of an object as a function of position on the object. With these methods, one must generally have an a priori knowledge of the vibration waveform (one usually assumes it is purely sinusoidal). Although this assumption is often fulfilled, there are definitely many cases where it is not valid, in particular, when the driving force itself is non-sinusoidal. In the vibration analysis method presented here, however, both the amplitude and the vibration waveform can be simultaneously measured; no such a priori knowledge or limiting assumption is necessary.

The vibration waveform is clearly an important parameter in vibration analysis. For example, the waveform has a definite effect on stress analysis calculations. The peak forces associated with a jerky, rectangular vibration waveform () will be much larger than those for a sinusoidal motion of the same amplitude. Furthermore, the vibration waveform will often vary from point to point on the object; further influencing stress calculations. Also, the vibration waveform may itself be of intrinsic value, such as in the design of loudspeaker diaphragms.

In section II we will briefly review the other existing holographic vibration analysis methods. This will give us an opportunity to present theoretical material shared by both existing methods and our method, and at the same time allow us to more clearly see the differences between the different approaches. In section III we will develop the theory of our multiplex

method, followed by our experimental results in section IV. Finally, in section V we will summarize the results, and try to objectively evaluate the advantages as well as weaknesses of our approach, and suggest possible areas for further development.

II A REVIEW OF HOLOGRAPHIC VIBRATION ANALYSIS METHODS

We assume that the reader is familiar with the basic principle of holographic wavefront reconstruction. Here we will only outline the basic properties of holography necessary for an understanding of the material which follows. For more detailed treatments of the fundamental principles involved, the reader may refer to the literature listed in the references [1 - 5].

The most common holographic method for vibration analysis is usually referred to as the time-average method. Its initial development was carried out in 1965 by Robert Powell and Karl Stetson at the University of Michigan [6, 7]. Since that time, many authors have published refinements and modifications of the basic technique, to suit particular applications; the present volume itself being one such modification. Other modifications will be discussed later.

Consider the system shown in fig. 1. By means of a suitable beam splitter and several mirrors, a laser beam is split up to form two mutually coherent beams; an object beam and a reference beam, directed as shown. The beams are then expanded by means of short focal length lenses (microscope objectives); the object beam illuminates an object (for the moment assumed to be stationary), while the reference beam directly illuminates a piece of high-resolution photographic film. If we denote the amplitude of the wave reflected by the object by the complex quantity U (hereafter referred to simply as the object wave) and the corresponding amplitude of the reference wave by R , then the intensity, I , recorded by the photographic film will be given by

$$I = \|U + R\|^2 = UU^* + RR^* + U^*R + UR^* \quad (1)$$

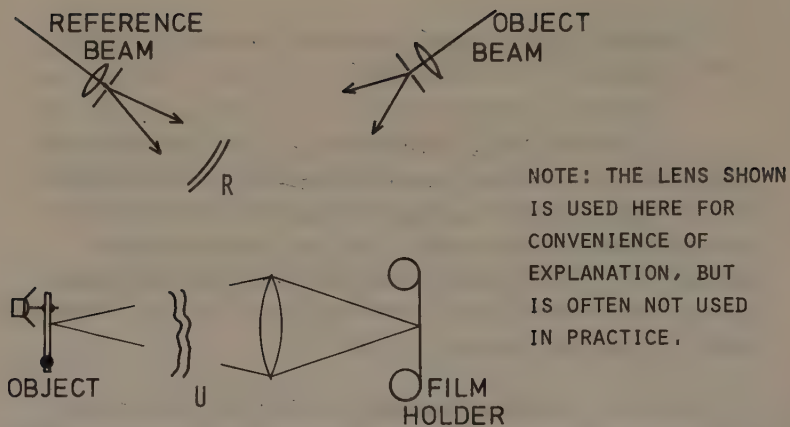


FIGURE 1: TIME-AVERAGE HOLOGRAM CONSTRUCTION

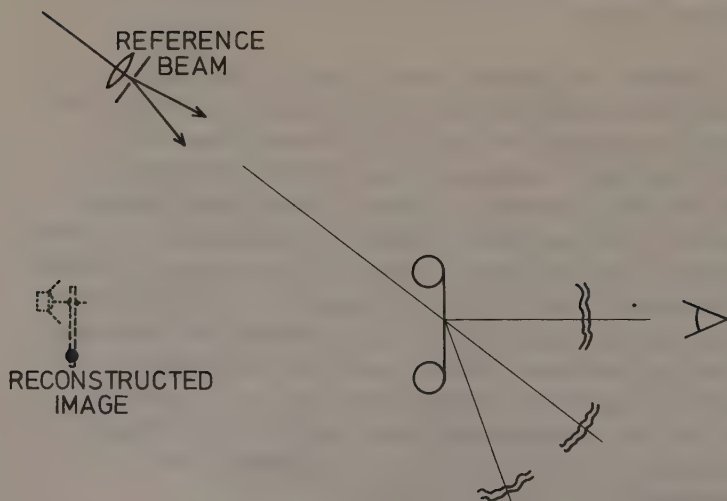


FIGURE 2: TIME-AVERAGE HOLOGRAM RECONSTRUCTION

Proper exposure and development of the photographic film will lead to a transmittance approximately proportional to I . (This approximation does not significantly effect our results. A more detailed description of holographic technique and the effects of film non-linearities can be found in the literature [1 - 5].)

Reillumination by the same reference wave R (fig. 2) will result in basically 3 waves propagating away from the hologram. The first two terms in eq. 1 contain typically low spatial frequencies, and result in basically undiffracted waves, as shown. The 3rd and 4th terms result from the interference between the object and reference waves. Because of the relatively large angle between these waves, these terms contain relatively high spatial frequencies; the 3rd term diffracting light away from the optical axis and the 4th term toward the axis. Clearly, if the intensity of the reference wave is uniform (i. e. $RR^* = \text{constant}$), then the 4th term gives us a good reconstruction of our original object wave U .

Let us now assume that our object is no longer stationary during the recording process. We must then write the intensity at the hologram as a function of time:

$$I(t) = \|U(t) + R\|^2 \quad (2)$$

The total exposure, for an exposure time τ , will be proportional to the average intensity:

$$I_{av} = \frac{1}{\tau} \int_0^{\tau} I(t) dt \quad (3)$$

If we substitute eq. 2 into 3 and retain only the 4th term in the resulting expansion, we obtain a reconstructed wave, U_R , given by:

$$U_R = \frac{1}{\tau} \int_0^{\tau} U(t) dt \quad (4)$$

(For convenience we have set $RR^* = 1$, and ignored unimportant scale factors.) What we have measured, in other words, is the average value of the object amplitude during the exposure time. Note the important difference here between the averaging process performed by a hologram, and that produced by other optical detectors such as the eye, photomultipliers, or conventional photography; all of which measure only average intensity.

Let us now restrict our object motion to small displacements along the bisector of the angle θ between the illumination wave and the viewing direction. We can then express our object wave as

$$U(t) \approx U e^{i\Phi(t)} \quad (5)$$

where U is the wave reflected from the stationary object and the phase Φ is given by:

$$\Phi(t) = \frac{2\pi}{\lambda} \cdot 2 \cos\left(\frac{\theta}{2}\right) \cdot Z(t) \quad (6)$$

where $Z(t)$ is the position of the object along the bisector mentioned above. ($Z(t)$ may also be a function of x and y)

If we further specialize to the case of sinusoidal motion:

$$\Phi(t) = A \sin(\omega t) \quad (7)$$

then our holographic reconstruction, eq. 4, gives

$$\begin{aligned} U_R &= U \cdot \frac{1}{\tau} \int_0^\tau e^{iA \sin(\omega t)} dt \\ &= U \cdot J_0(A) \end{aligned} \quad (8)$$

where J_0 is the zeroth order Bessel function of the first kind. (The above (eq. 8) is exact for τ an integral

number of oscillation periods, and approximately true for exposure times which are long compared with the exposure period.) Thus, on reconstruction we see our original object, U , and the intensity of the reconstructed image will be proportional to $J_0(A)^2$. A somewhat more detailed derivation of this time-average effect may be found in reference 6.

This result is consistent with intuition: To each object point there corresponds a set of fringes in the hologram plane; motion of the object point leads to motion of the fringe system, which, in turn, leads to a (blurred) fringe system with reduced contrast; the reduced contrast of this fringe system will lead to reduced intensity upon reconstruction. We can carry this argument one step further. If we note that the photographic process performs an incoherent (sequential) integration of fringe systems, we see that the time averaging effect of the holographic process is equivalent to multiplying the original object amplitude by the optical transfer function associated with the object motion, as in eq. 8. (This last comment may be useful for those familiar with the concept of optical transfer function, but we will not base any conclusions on this viewpoint.)

Experiments illustrating the above theory were carried out in our laboratory. The object (fig. 3) consisted of a flat plate of aluminum, pivoted on two points near its left edge, and coupled to a loudspeaker on the right. This provided a linear amplitude vs position relationship, making comparison of the results with theory easier. The plate surface was diffusely reflecting (painted with flat white paint).

Typical results are shown in figures 4 and 5. The $J_0(A)$ dependence is quite obvious, and the amplitude of vibration can be easily measured. The right edge of the plate lies at the fourth zero; reference to tables indicates that the fourth zero of $J_0(x)$ lies at $x = 11.8$; use of eq. 6 gives $A \approx 480 \text{ nm}$ ($\lambda = 5145 \text{ \AA}$).



FIGURE 3: THE OBJECT

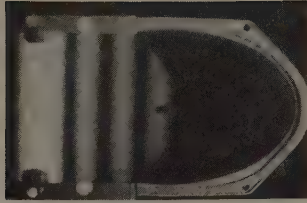


FIGURE 5: SINUSOIDAL
VIBRATION

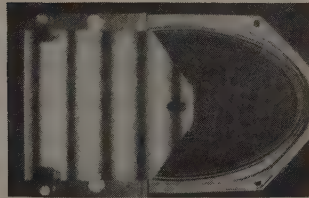


FIGURE 6: SQUARE-WAVE
VIBRATION

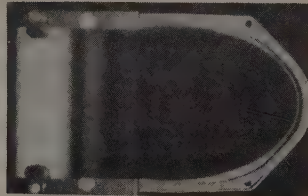


FIGURE 7: TRIANGULAR
WAVE VIBRATION

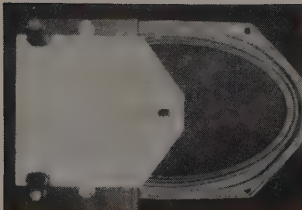


FIGURE 4: HOLOGRAPHIC
RECONSTRUCTION, NO
VIBRATION PRESENT

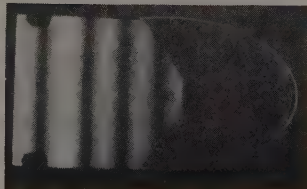


FIGURE 8: SINUSOIDAL
VIBRATION WITH PHASE
SHIFTED REFERENCE WAVE

It should be emphasized that one obtains a J_0 fringe profile only in the case of sinusoidal vibration. Other vibration waveforms will lead to, in general, different fringe profiles, as determined by eq. 4. For example, evaluation of eq. 4 for a square-wave vibration (□□□) yields a fringe profile given by $\cos(A)$; the same evaluation for a triangle wave (WWW) gives a $\frac{\sin(A)}{A}$ fringe profile. The corresponding holographic results are shown in figures 6 and 7. The peak amplitude, A , was the same in all three cases (figures 5, 6, 7). The agreement with theory is quite good. Note, for example, the rapid falloff of fringe visibility in the triangle wave case. A more detailed comparison between experiment and theory will be given in section IV.

Many modifications to the above technique have been suggested. One class of modifications which is of particular interest to us involves introducing a time-dependent phase shift between the object and reference waves [8 - 13]. For example, a phase shift which is linear in time (i. e., a frequency shift) can be easily obtained by passing one of the beams through a moving grating. If we use the first diffraction order diffracted from a grating of period d , moving with velocity v transverse to the beam axis, then the frequency shift $\Delta\omega/2\pi$ introduced will be $\frac{v}{d}$ Hz.

The effect this has on a time average vibration hologram can be readily calculated. The phase factor in eq. 7 must be modified since only the phase difference between object and reference waves is important:

$$\Phi(t) = A \sin(\omega t) - 2\pi \frac{v}{d} t \quad (9)$$

If we adjust the velocity of our grating so that $2\pi v/d = \omega$, then the holographic reconstruction eq. 8 becomes:

$$\begin{aligned} U_R &= U \cdot \frac{1}{\tau} \int_0^\tau e^{i[A \sin(\omega t) - \omega t]} dt \\ &= U \cdot iJ_1(A) \end{aligned} \quad (10)$$

As before, the exposure τ is an integral number of oscillation periods. A reconstruction of a hologram made in this way is shown in fig. 8. Notice the $J_1(A)$ fringe profile. Also, since $J_1(0) = (0)$, the stationary parts of the object are not visible. The primary advantage of this modulation technique lies in its ability to detect small vibration amplitudes. The presence of a small vibration during recording results in reconstruction, in a departure from zero visibility (instead of a departure from unit visibility, in the unmodulated approach). With other forms of modulation, the range of vibration amplitudes measurable with time-average can be extended to larger amplitudes [8, 10]. The modulation technique can also be used to map the phase of a sinusoidal vibration, as well as measure amplitude [11]. By modulating the laser intensity, the contrast of the vibration fringes can be increased, thereby making higher order fringes more visible [14, 15]. The measurement range can also be extended by vibrating the holographic plate in synchronism with the object [16]. The accuracy of vibration fringe location measurement can be greatly increased by electronic detection methods; accuracies to 6×10^{-4} fringe have been reported [17]. There have been modifications proposed to optimize the recording time, generally involving pre-exposure of the holographic plate [18 - 20]. Another multiple exposure method is holographic subtraction, aimed at measuring small vibration amplitudes [21, 22]. Holographic vibration fringes can also be observed in real time [23].

Another common family of holographic vibration measurement methods consists of stroboscopic methods [24 - 31]. Although these methods tend to be more complex than normal time average methods, they usually offer higher contrast fringes. A comparison of stroboscopic and time average methods can be found in reference [32].

Several methods have been proposed for measuring the phase of a sinusoidal, multi-mode vibration (time-dependence is sinusoidal, but amplitude and phase depend

on position) [11, 33, 34]. Methods involving cinematography have been proposed for the measurement of non-sinusoidal vibrations, such as relatively slow transient phenomena [35]. Theoretical work has been done on the application of time average holography to the study of decaying sinusoidal vibrations [36]. Little work has been done to date on the problem of holographically measuring the vibration waveform itself. An interferometric method involving electronic detection has been developed for waveform measurement [37]. The method offers quite good results, but is applicable to only a restricted class of objects (specular reflectors). The method we describe in this volume is applicable to specular or diffuse, transmitting or reflecting objects.

Finally, some examples of applications of holographic vibration measurement are given in references [38, 39]. (The objects investigated were ship propellers and aircraft structures.)

III THEORETICAL DEVELOPMENT OF THE MULTIPLEX APPROACH

A) The Object Motion - Object Wave Transformation

Consider the object wave introduced in section II:

$$U(x,t) = e^{i\Phi(x,t)} \quad (11)$$

where, as explained in section II, $\Phi(x,t)$ is a real function describing the motion of our object.

A (two-dimensional) position coordinate, x , in the argument of Φ , allows for the possibility that the vibration amplitude and waveform may vary from point to point on the object. In eq. 11 we have, for simplicity, assumed that the object wave, $U(x,t)$, has unit amplitude. A static amplitude factor can be included later if desired without affecting the results in any significant way.

If Φ is periodic in time, then we may expand it in a temporal Fourier series:

$$\Phi(x,t) = \sum_{-\infty}^{\infty} A_m(x) e^{im\omega t} \quad (12)$$

where ω is the fundamental frequency of the mechanical vibration. (In a later section we will consider the inclusion of non-harmonic frequencies in the expansion of Φ .) The coefficients $A_m(x)$ in the above series can be, in general, complex, but will always appear in complex conjugate pairs ($A_m = A_{-m}^*$) so that the resulting function $\Phi(x,t)$ will be real. The coefficient A_0 describes the (static) surface profile of the object.

The wave reflected by our object, $U(x,t)$, can also be expressed as a Fourier series:

$$U(x,t) = \sum_{-\infty}^{\infty} U_n(x) e^{in\omega t} \quad (13)$$

With our multiplex holographic method, we measure the coefficients $U_n(x)$. From these measurements we reconstruct the object motion function $\Phi(x,t)$.

In general, the coefficients $A_m(x)$ and $U_n(x)$ are related in a rather complicated fashion. For instance, in the sinusoidal case:

$$\Phi(x,t) = A(x) \sin(\omega t) \quad (14)$$

we have:

$$U_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\Phi(x,t)} e^{-in\omega t} d(\omega t) \quad (15)$$

$$= J_n(A(x)) \quad (16)$$

Conventional time-average holography, as discussed in section II, thus measures only the zeroth temporal Fourier component of the object wave, U_0 . In our multiplex holographic method (to be described in detail later), we simultaneously record both the amplitudes and phases of many of the coefficients U_n , thus giving us much more complete information about the object motion.

We can use eq. 15 to easily calculate the coefficients corresponding to the simple waveforms considered in section II:

<u>Vibration Waveform</u>	<u>Coefficients</u>
sinusoid	$U_n = J_n(A)$
square wave	$U_n = \text{sinc}\left(\frac{n}{2}\right) \cos\left(A + \frac{n\pi}{2}\right)$
triangular wave	$U_n = \frac{1}{2} \left[\text{sinc}\left(\frac{A}{\pi} + \frac{n}{2}\right) + (-1)^n \text{sinc}\left(\frac{A}{\pi} - \frac{n}{2}\right) \right]$

As before, A is the peak phase displacement (related to the peak mechanical displacement by eq. 6). Also, $\text{sinc}(x) \equiv \frac{\sin(\pi x)}{\pi x}$.

There are alternative methods to calculate these coefficients. Results can often be quickly obtained using probability arguments. This method is discussed in Appendix 1. Also, the coefficients can be expressed as expansions of Bessel functions. This approach is discussed in Appendix 2.

Notice that even for an object vibrating at only a single frequency, the resulting reflected wave $U(x,t)$ contains all frequencies (from eq. 16).

B) The Reverse Transformation

Our multiplex holographic method measures the coefficients $U_n(x)$. Of mechanical interest however, is the vibration function Φ (or equivalently, the Fourier coefficients A_n). Thus we need a method to calculate the waveform Φ from knowledge of the coefficients U_n .

One way for doing this is to simply use the definition of U in terms of Φ (eq. 11). From knowledge of the U_n , we can first construct $U(x,t)$ by use of eq. 13. The vibration waveform Φ can then be calculated by taking the logarithm, since from eq. 11 we have:

$$\Phi(x,t) = -i \ln U(x,t) \quad (17)$$

From the function Φ calculated in eq. 17 we may readily calculate the Fourier coefficients A_n , if desired.

Another method for converting our knowledge of the U_n into information about the vibration waveform Φ has been developed in our laboratory [40]. The approach used is to compare temporal derivatives:

$$U(x,t) = e^{i\Phi(x,t)}$$

$$\frac{\partial U}{\partial t} = i \frac{\partial \Phi}{\partial t} e^{i\Phi}$$

or

$$U^* \frac{\partial U}{\partial t} = i \frac{\partial \Phi}{\partial t} \quad (18)$$

If we now express both sides of eq. 18 as a Fourier series, we have the desired result:

$$\begin{aligned} U &= \sum U_n e^{in\omega t} \\ U^* \frac{\partial U}{\partial t} &= \sum_n \sum_j in\omega U_n U_j^* e^{i(n-j)\omega t} \\ &= \sum_m \sum_n in\omega U_n U_{n-m}^* e^{im\omega t} \\ i \frac{\partial \Phi}{\partial t} &= i \sum_m im\omega A_m e^{im\omega t} \\ A_m &= -\frac{i}{m} \sum_n n U_n U_{n-m}^* \end{aligned} \quad (19)$$

Notice that the static phase, A_0 , if desired for say, contouring purposes, must be measured by a separate experiment; eq. 19 is not suitable.

Clearly, the two methods presented above should yield the same result, at least in the absence of noise in the measurement of the U_n . Although the first method

is perhaps conceptually more direct, the second method may be simpler and quicker in practice, since it involves manipulation of only a few numbers (the most significant U_n), instead of the manipulation of functions.

C) The Holographic Measurement Process

We assume for the moment that our object vibration is periodic and, as before, we expand our object wave in a temporal Fourier series:

$$U(x, t) = \sum U_n(x) e^{in\omega t} \quad (20)$$

where

$$U_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} U(x, t) e^{-in\omega t} d(\omega t) \quad (21)$$

A multiplex hologram is made of the object with the experimental arrangement shown in fig. 9. In the recording process, the wave reflected from the object is imaged through a small aperture onto a moving grating. The amplitude transmittance of the grating can be expressed as:

$$G(x - vt) = \sum C_m e^{2\pi i m v_0 (x - vt)} \quad (22)$$

where v_0 is the spatial frequency of the grating. During the recording process, the grating velocity v is synchronized with the fundamental vibration frequency ω according to:

$$2\pi v_0 v = \omega \quad (23)$$

(Motion of one grating period $d = \frac{1}{v_0}$ during one oscillation period $\frac{2\pi}{\omega}$.)

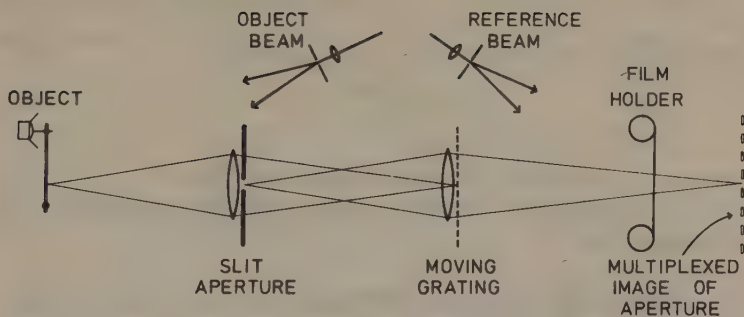


FIGURE 9: MULTIPLEX HOLOGRAM RECORDING

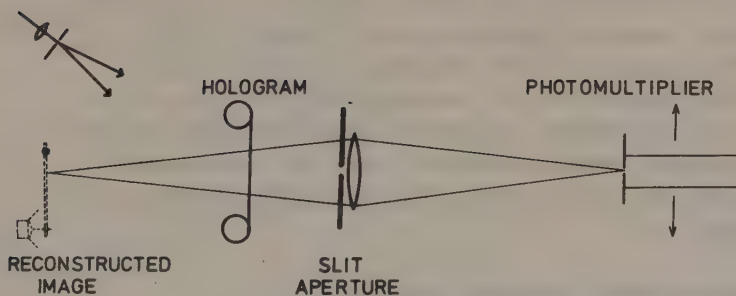


FIGURE 10: MULTIPLEX HOLOGRAM RECONSTRUCTION

The desired image recorded on the holographic plate will be the average value of the product of $U(x,t)$ and $G(x,t)$ during the exposure time τ :

$$\langle UG \rangle = \sum_m C_m e^{2\pi i m v_0 x} \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} U(x,t) e^{-i m \omega t} dt \quad (24)$$

(This is eq. 4 with $U(t)$ now generalized to $\langle UG \rangle$)

If τ is an integral number of vibration cycles, then eq. 24 reduces to:

$$\langle UG \rangle = \sum_m C_m U_m(x) e^{2\pi i m v_0 x} \quad (25)$$

Thus many of the coefficients U_n are recorded as a multiplex hologram, with each coefficient U_n riding on a separate spatial carrier frequency, nv_0 .

The reconstruction arrangement is shown in fig. 10. The slit aperture in fig. 10 is located in the same plane as the multiplex image of the aperture in fig. 9. By varying the location of the slit aperture in reconstruction, one may select different coefficients U_n to study. These separate images, U_n , may be imaged directly onto a photomultiplier, or photographed by placing a camera directly behind the slit aperture. The coefficients U_n may also be combined to form an image displaying various other object motion parameters, such as velocity and acceleration distributions. This is discussed further in appendix 3.

D) Non-Harmonic Vibration Analysis

In eq. 20 we assumed our object was periodic, with fundamental frequency ω . We now consider nonharmonic frequencies in the spectrum of Φ .

Assume our object is vibrating at a single frequency $\beta\omega$, where ω is the frequency defined by our moving grating,

and β is in general not an integer. Without restricting the validity of the final result, we can consider (for mathematical convenience) the function Φ to be a phasor rotating in the complex plane. Thus, the function Φ can be written as:

$$\Phi = A e^{i\beta\omega t} \text{rect}\left(\frac{t}{\tau}\right) \quad (26)$$

$$\text{where } \text{rect}\left(\frac{t}{\tau}\right) \equiv \begin{cases} 1, & |t| \leq \frac{\tau}{2} \\ 0, & |t| > \frac{\tau}{2} \end{cases}$$

(In any actual oscillation, the frequencies $+\beta\omega$ and $-\beta\omega$ will occur as conjugate pairs, so that the total phase function Φ is real. At this stage we consider one component of such a pair). We explicitly include the $\text{rect}\left(\frac{t}{\tau}\right)$ factor in eq. 26 to emphasize that the measurement process is finite. The function defined in eq. 26 is not periodic. However, we may still expand Φ in a Fourier series since the function Φ in eq. 26 is zero outside of the exposure time τ . Hence we are able to generalize the algorithm presented above (eq. 19), based on the properties of Fourier series.

Thus we expand eq. 26 in a Fourier series:

$$\Phi_o(x, t) = A \sum_{-\infty}^{\infty} A_p e^{ip\omega t} \quad (27)$$

$$\begin{aligned} A_p &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\beta\omega t} e^{-ip\omega t} d(\omega t) \\ &= \text{sinc}(\beta - p) \end{aligned} \quad (28)$$

The expansion in eq. 27 is valid only in the interval $-\pi \leq \omega t < \pi$. Outside this interval, the series converges to the periodic continuation of Φ .

A hologram can now be made of the object as outlined in the previous section, with the additional constraint that the duration of the exposure should equal the time it takes the grating to move through one cycle. The coefficients can then be measured and the algorithm, eq. 19, used to compute the desired amplitude:

$$A \operatorname{sinc}(\beta - p) = -\frac{i}{p} \sum_n U_n U_{n-p}^* \quad (29)$$

If the frequency $\beta\omega$ is known, then evaluation of the right hand side of eq. 29 for any non-zero p will yield the desired amplitude. (The value of p closest to β will give the maximum coefficient A_p and hence probably be optimum in terms of signal to noise). If the frequency $\beta\omega$ is not known, then the right hand side of eq. 29 must be evaluated for at least two values of p , and the resulting equations solved for the amplitude and β .

The above results can be easily extended to the case of exposure of N cycles of grating motion. By changing the limits of integration in eq. 28, we find:

$$A_p = \operatorname{sinc}[(\beta - p)N]$$

If there are several non-harmonic frequencies present the algorithm (eq. 29) can be generalized as follows:

$$\sum_m A_m \operatorname{sinc}(\beta_m - p)N = -\frac{i}{p} \sum_n U_n U_{n-p}^* \quad (30)$$

This result is valid for integral as well as non-integral values of β , since $\operatorname{sinc}[(\beta_m - p)N] = \delta_{\beta_m, p}$ for β_m an integer.

If N is made large, then only integral or very nearly integral values of β_m will contribute to the sum on the left hand side of eq. 30. Hence a long exposure multiplex hologram will allow us to analyse the harmonic content of our object motion without interference from the non-harmonic

terms. If then a short exposure hologram is made, the harmonic content can be easily subtracted from eq. 30, making separation of the non-harmonic terms simpler.

Eq. 23 can also be used to determine how good our synchronism of the grating motion and object frequency must be in the harmonic case. The argument of the sinc function, $(\beta - p)N$, is equal to the amount of phase drift in cycles, between our object frequency $\beta\omega$ and the frequency $p\omega$ (defined by the grating), during the exposure time τ . Hence, if the relative phase drifts by one quarter cycle, the measured amplitude will drop by 10 %. If the amount of phase drift is known, then the results can be corrected using this sinc factor. If, however, the drift is unknown, we can conclude that the drift should be less than one quarter cycle in order to expect the results to be valid to within 10 %.

IV EXPERIMENTAL VERIFICATION

The experimental verification of the multiplex approach was undertaken in two stages. In the first set of experiments, measurements were made of the intensities of the various coefficients, $|U_n|^2$, as a function of vibration amplitude, for various vibration waveforms. For the second set of experiments, the apparatus was refined and developed to measure the phase of the coefficients U_n ; several vibration waveforms were studied and the results compared with theory.

A) Intensity Measurements of the Coefficients, $|U_n|^2$

1. The Experimental Setup

A diagram of the apparatus used is shown in figures 9 and 10. The object used for the simple time-average holograms (described in section II) was rebuilt, with the driven end of the plate sandwiched between stiff springs. This increased the fundamental resonant frequency of the object from about 25 Hz to about 150 Hz, thereby insuring that the object would accurately follow the input waveform (fundamental frequency ~ 10 Hz).

The object was driven directly by a Krohn-Hite® 5100 A signal generator. The grating used was an inexpensive Ronchi ruling (a grating of alternating transparent and opaque regions of equal width) with 200 lines per inch, purchased from Edmunds Scientific®. From eq. 25 it is clear that it is desirable to use a diffraction grating with approximately uniform diffraction efficiency in at least the first 5 or so diffraction orders; where most of the information about U is likely to be found. From theoretical considerations, then, a Ronchi ruling would seem to be a very poor choice, since (in theory) a Ronchi ruling will not diffract any light

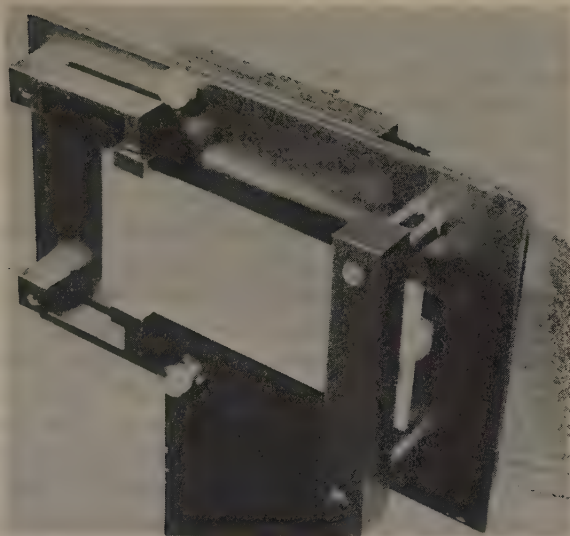


FIGURE 11: THE OBJECT FOR INTENSITY MEASUREMENTS

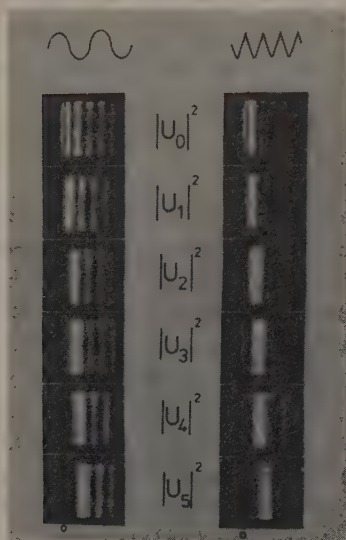


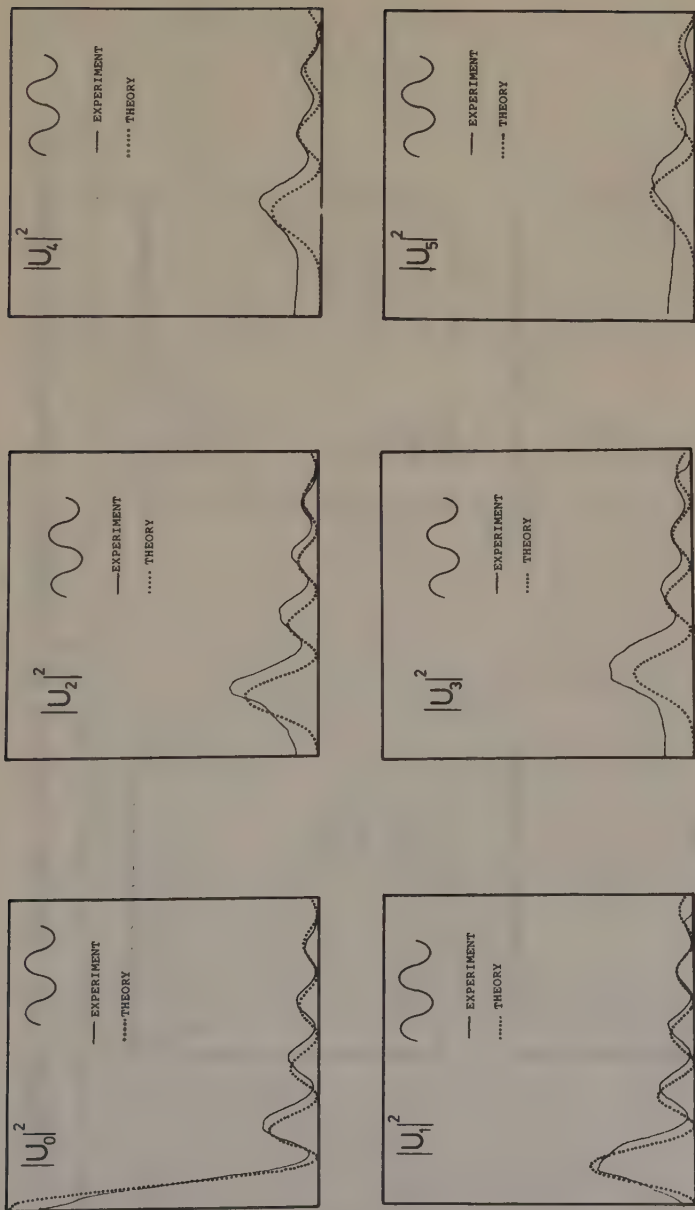
FIGURE 12: RESULTS FOR SINUSOIDAL AND TRIANGULAR WAVEFORMS

into non-zero even orders (i. e. $C_m = 0$ for $m = \text{non-zero, even}$). In other words, the use of a Ronchi ruling would mean all information about the U_n for all even, non-zero n would be lost. Fortunately (for this application) it is very difficult (expensive) to produce a Ronchi ruling which behaves according to simple diffraction theory, since a very slight phase nonuniformity in the transparent regions of the grating will lead to the appearance of even orders. Indeed, with the Edmunds[®] grating used, the diffraction efficiency was almost a monotonic slowly decreasing function of order number, with the even orders being almost as strong as the neighbouring odd orders.

The 5145 Å line of a Spetra Physics[®] argon ion laser was used. The illumination beam was approximately collimated by a simple lens (not shown in fig. 9), increasing the illumination intensity at the object and thereby reducing exposure time. Exposure times of about 1 second were used, utilizing a laser power of about 100 mW. The film was Agfa[®] 10E56, in 35 mm format, with conventional processing. After processing, the film was returned to the film holder, and reilluminated by the reference beam. The reconstructions were photographed, and also directly scanned by a photomultiplier, as shown in fig. 10.

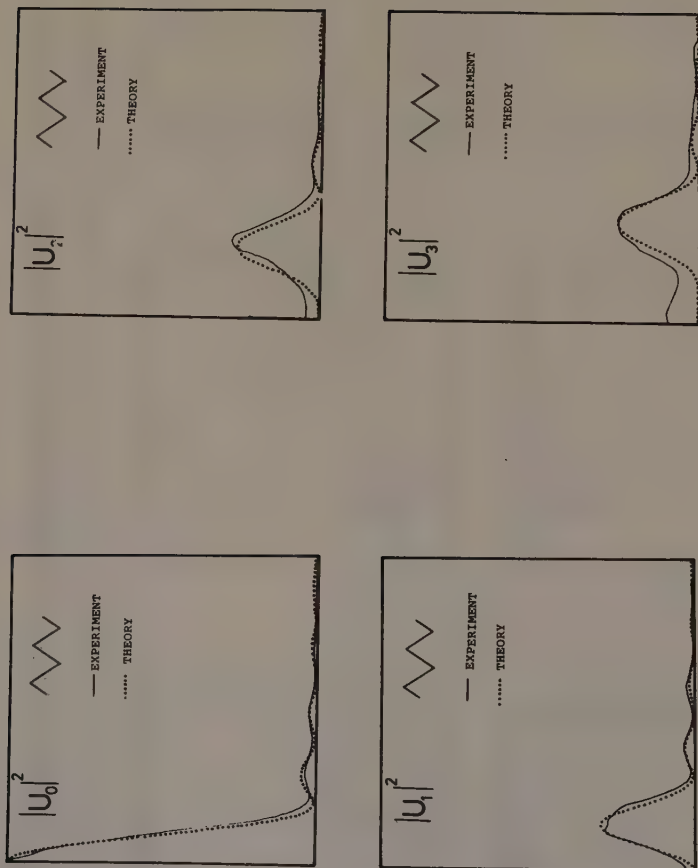
2. The Results

The photographic results are shown in fig. 12. The frame of the object was not illuminated during the hologram recording, and hence only the plate itself appears in the reconstructions. The dark line near the left of each column indicates the location of zero vibration; the amplitude of vibration varies linearly with distance from this line. The amplitude of vibration at the right edge of the reconstructions was about 600 nm ($\sim 1 \lambda$). It should be emphasized that for a given waveform, all of the reconstructions shown were obtained from a single hologram; during reconstruction the location of the slit in fig. 10 was varied to photograph reconstructions for



Note: Horizontal axis is vibration amplitude, a . (Left edge $a=0$; right edge $a \approx 600$ nm)

FIGURE 13a: PHOTOMULTIPLIER MEASUREMENTS —SINUSOIDAL VIBRATION



Note: Horizontal axis is vibration amplitude, a . (Left edge $a=0$; right edge $a \approx 600$ nm)

FIGURE 13b: PHOTOMULTIPLIER MEASUREMENTS -TRIANGULAR VIBRATION

different values of n . The results of the photometric scans of the reconstructions are shown in figures 13 a, b. For low values of n the agreement with theory is quite good. For higher n values, background noise appears. The poorer diffraction efficiency of the grating at higher n values reduced the reconstruction intensity considerably. The background noise seen in the reconstructions for larger n values appears to be weighted towards the left side of the reconstruction (the location of zero amplitude and the very strong peak in U_0). Film grain noise is probably scattering some of the energy from this bright peak in U_0 in the directions corresponding to higher n values.

Reconstructions were visible to the eye out to about the 12th order. The background noise for such large n was, however, very large.

On the basis of results such as those in figures 12 and 13 a, b it seems clear that the multiplex approach can definitely yield useful information about the higher order temporal Fourier coefficients of the object wave U ; especially if steps are taken to reduce the background noise, or if some sort of averaging methods are used to subtract the effect of the background.

B) Phase Measurements and Vibration Waveform Reconstruction

The phase measurements were made using a technique known as real-time holographic interferometry. In this technique, both the hologram and the object are reilluminated after development of the hologram.

If the system was not disturbed in any way during the time between the hologram recording and reconstruction processes, then the holographically reconstructed image of the object will coincide with the object itself, and any subsequent deformation of the object will result in interference fringes being seen. (For a more detailed explanation of real-time holographic interferometry see [4, 41, 42].) Any optical system used for real-time holographic interferometry must be extremely stable,

since any relative motion of the components (even as small as a wavelength) will result in generally undesirable fringes visible in the reconstruction.

1. The Experimental Setup

Because of the strict stability requirement, mentioned above, it was necessary to improve the mechanical stability of several components in the system. The Object was mounted more rigidly (fig. 14) and a stable holographic plate holder was built. The plate holder (fig. 15) was designed so that the plates could be developed without removing them from the holder. The final setup is shown in fig. 16. The arrangement is identical to that of figures 9 and 10, except for the addition of a thin glass plate in object beam (used as a phase shifter as explained later).

The initial measurements with the system were plagued by a high-background problem, the source of which was traced to the orientation of the hologram plate. It is common practice in real-time holographic interferometry to orient the hologram so that the normal to the plate's surface lies parallel to the bisector of the angle between the direction of propagation of the reference beam and the direction of propagation of the object beam. (With this geometry, the holographic interference fringes have less of a tendency to shift when the photographic emulsion shrinks during the fixing process, since the fringes are distributed throughout the emulsion thickness perpendicular to the plate surface. Any shift in the fringes in the hologram can cause undesirable effects in the reconstruction process.) When this orientation was used in our case, however, more problems were created than were solved, since the hologram plate then reflected part of the reilluminating reference wave back along the axis of the interferometer, where it

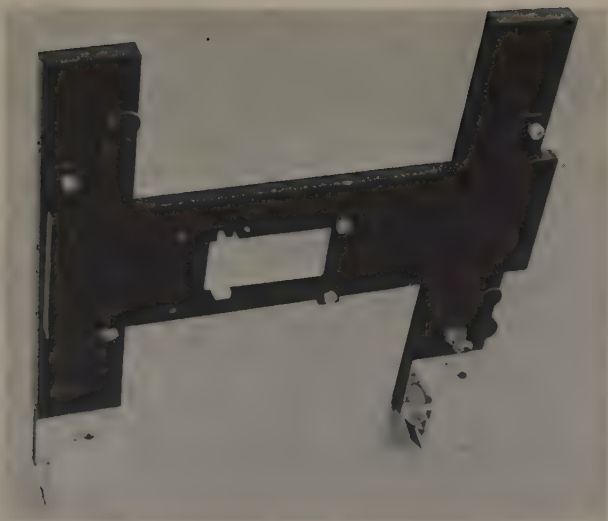


FIGURE 14: THE OBJECT USED FOR PHASE MEASUREMENTS



FIGURE 15: THE HOLOGRAPHIC PLATE HOLDER



FIGURE 16: THE HOLOGRAPHIC INTERFEROMETER



FIGURE 17: THE DETECTOR END OF THE INTERFEROMETER

was again reflected off of the various lenses, etc., finally propagating back through the detecting slit and into the photomultiplier; creating a spurious signal as much as 100 times larger as the desired signal!

The optics for the detection system (the aperture, lens, and photomultiplier shown in fig. 10) were mounted on a moveable steel beam (the wide black stripe in fig. 16). The beam was pivoted about a point directly under the reconstructed image of the object (also the location of the moving grating). By sliding the beam from side to side, different values of n could be obtained (because of the motion of the aperture), while the location of each image U_n , relative to the photomultiplier, remained unchanged since the lens and photomultiplier moved as a unit. (In theory, only the aperture should have to be moved. However, a lens large enough to pass all of the U_n (without repositioning or vignetting) was not available.) Another view of the detection end of the setup is shown in fig. 17.

An Alpha-16 computer from Computer Automation[®] was used for the electronic control of the experiment. During the recording process, the computer performed 3 tasks: the control of the exposure shutter, the control of the translation stage for the moving grating, and the driving of the object with the desired waveform. The computer first calculated and stored in memory (in a 400 point matrix) the desired waveform. The output of the data was controlled by an external signal generator. When the computer received a pulse from the external generator, the next point on the waveform was sent to a digital-analog converter, which in turn drove the loudspeaker. By counting the number of data points, the computer also opened the exposure shutter for a predetermined number of vibration periods, and also sent pulses to the stepping motor on the translation stage, thereby maintaining synchronism between the object and the grating, as required by eq. 23.

This computer-controlled system proved to be very flexible, and solved many control problems. Any arbitrary waveform could be used, and the temporal frequency of the vibration could be easily changed at any time by changing the frequency of the external signal generator (without disturbing the synchronism between the moving grating and the object vibration). During the reconstruction process, the computer was used to calculate the U_n (in amplitude and phase) from the photomultiplier measurements, and to compute and plot the waveform Φ , using the algorithm (eq. 19).

2. Hologram Recording

The peak amplitude of the object vibration used was very small ($\sim \frac{1}{20} \lambda$). This was necessary in order to prevent the information about U from spreading into higher orders, where the signal-to-noise ratio is poorer. (A discussion of this small vibration amplitude requirement is given in appendix 4.) An exposure of 20 erg/cm^2 was found to give good results with Agfa® 10 E 56 photographic plates. Exposure times were typically 5 seconds, requiring 500 mW of laser power (argon ion, $\lambda = 5145 \text{ \AA}$). The object vibration frequency was about 1 Hz, but was adjusted as needed to give the proper exposure in exactly 6 vibration periods. The plates were then processed by raising appropriate containers of chemicals up around the plates; the containers were supported by a magnetic stirrer, thereby providing slight agitation during the development process. The processing was normal: 5 minute develop in Ultra-fin® developer, diluted 1 : 10, followed by a 4 minute fix, 3 minute wash, and air-dried for about 10 minutes before measurements were begun. (Experience showed that waiting longer than 10 minutes produced no significant change in the stability of the subsequent measurements.) Because 10 E 56 plates are insensitive to red light, it was possible to cut, expose, and develop the plates with a red safe-light always on, thereby eliminating the frustrations caused by knocking something over in the dark, or fixing instead of developing, etc.

During both the exposure of the plate and the measurements following development, the entire interferometer was enclosed in a relatively air-tight and light-tight wooden box, thereby reducing the undesirable effects caused by stray light and air turbulence.

3. The Phase Measurement Process

For a given position of the aperture in fig. 10, there are in general two images superimposed on the photomultiplier. One image is the reconstruction of U_n from the hologram (the value of n being determined by the location of the aperture). The other image corresponds to the wave which is reflected from the (now stationary) object, diffracted into the n^{th} order by the (now stationary) grating, and subsequently imaged onto the photomultiplier.

Because the two images are mutually coherent they will interfere, allowing us to derive phase information from intensity measurements. By means of simple electrical shutters in the object and reference beams, it was possible to select either one or both of these images for measurement.

If one measures the intensities of each of the images alone, the intensity of the superposition (sum) of the two, and the intensity of the sum obtained after shifting the phase of the object beam (or reference beam) by 90° , then a simple calculation (appendix 5) shows that the phase angle β , between the stationary object wave and the reconstructed U_n , can be obtained by

$$\tan \beta = \frac{I_3 - I_1 - I_2}{I_4 - I_1 - I_2} \quad (31)$$

where I_1 and I_2 are the intensities of the individual waves, and I_3 and I_4 are the intensities of the two sums mentioned above.

The phase of the object wave was changed as needed by means of the glass plate shown in fig. 16. By rotating the glass plate through small angles about a vertical axis it was possible to accurately shift the phase of the object beam by any desired amount. (The phase shifter was calibrated by observing the motion of real time holographic interference fringes.)

The interferometer housing was designed so that the phase shifter and the photomultiplier could be moved without opening the enclosure.

4. The Results

The intensities of the coefficients $|U_n|^2$ were very stable, and the measurements were reproducible. The phases, however, were still somewhat unstable, despite the precautions taken. If the photomultiplier was not moved, it was found that the phase of a typical coefficient varied approximately linearly with time; changing about 10° per minute. When, however, the photomultiplier was moved and the phases of all the U_n measured repeatedly in sequence, the results were much more erratic.

The phases were measured 8 times in sequence for n values ranging from -4 to $+4$. Instead of simply averaging the results for each U_n , it was found better to first calculate the phase differences between neighbouring U_n in each sequence, and take the average value of these differences in the 8 sequences. This averaging method produced an r. m. s. deviation in the phase of about 10° ; significantly smaller than that obtained by direct averaging, since the stability of the phases during the short time interval between neighbouring U_n measurements in a sequence was much better than the stability over the (relatively) longer time interval between sequences.

The amplitudes of the U_n were calculated from the intensities of the reconstructions. A value for the

background was obtained by measuring the intensity of the U_n for values of $|n|$ greater than 4; For these n , the reconstruction intensity was approximately independent of n , and was therefore assumed to be noise. This background was subtracted from all the intensities $|U_n|^2$.

From eq. 25 it is clear that we also need the diffraction coefficients C_m of the grating. This information is provided by the intensity measurements of the object wave alone in the real-time interferometer, since the object wave used in the interferometric measurements is also diffracted by the grating.

Two waveforms, Φ_1 and Φ_2 , were studied; both with two Fourier components. The following results were obtained:

$$\Phi_1 = \frac{\pi}{5} \left[\sin(\omega t) + (0.5) \sin(2\omega t + 45^\circ) \right]$$

n	Amplitude U_n	Phase difference between U_n (degrees)
- 4	0.12	
- 3	0.30	58
- 2	0.34	- 157
- 1	1.05	30
0	1.46	- 189
1	0.72	- 28
2	0.56	25
3	0.24	- 17
4	0.10	43

$$\Phi_2 = \frac{\pi}{5} \left[\sin(\omega t) + (0.4) \sin(3\omega t + 30^\circ) \right]$$

n	Amplitude U_n	Phase difference between U_n (degrees)
- 4	0.12	185
- 3	0.21	- 71
- 2	0.17	91
- 1	0.56	- 189
0	0.89	3
1	0.52	- 40
2	0.19	66
3	0.19	20
4	0.08	

The algorithm (eq. 19) was used to calculate the waveform from the above coefficients. The results are shown in figures 18 and 19.

The quality of the results depends on the type of information one wishes to derive from them; we have compared the results with theory from several viewpoints:

	Φ_1	Φ_2
R. M. S. deviation of the object trajectory from theoretical trajectory (expressed as fraction of R. M. S. amplitude)	11.3 %	15.4 %
Peak velocity reached by the object (compared with theory)	26.1 % too high	16.7 % too high
Peak acceleration reached by the object (compared with theory)	46.4 % too high	18.6 % too high

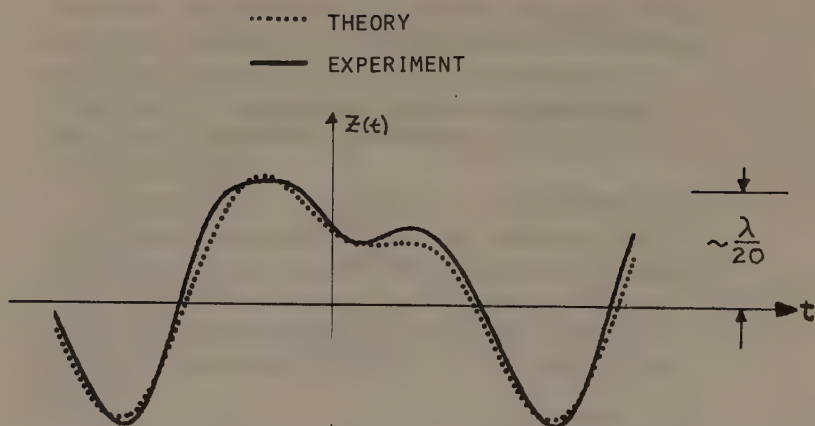


FIGURE 18: RESULTS FOR ϕ_1

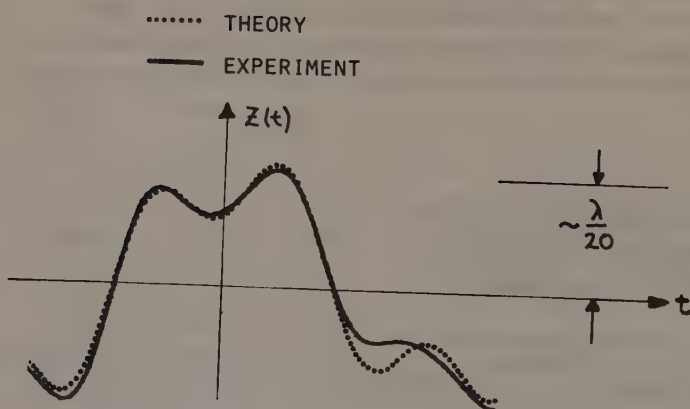


FIGURE 19: RESULTS FOR ϕ_2

From the above comparisons, we see that if the waveform itself is of primary interest, then the technique, in its present state of development, can offer reasonable (but admittedly not outstanding) results. If the peak shear force (hence peak acceleration) is important, then clearly the technique would need improvement. (The increase in error associated with the derivatives above is intuitively to be expected, since the differentiation process places more emphasis on the higher harmonics, where the signal-to-noise ratio is poorer.)

There are basically two areas where additional work could improve the above results. The signal-to-noise ratio of the hologram itself could possibly be improved, by special development, exposure tricks, etc. With the holograms used above, the signal-to-noise ratio was about 600 (noise being defined here as the background measured as described above). This figure is already quite good, and it is unlikely that any sweeping gains can be easily made within the state-of-the-art as it exists today.

The other area for improvement is environmental stability, in order to improve the phase measurements. Temperature stabilization, humidity stabilization, and more baffles to cut down on turbulence in the interferometer would all help. Also, a matrix of photo-multipliers allowing one to simultaneously measure many U_n would significantly alleviate the phase drift problem.

V CONCLUSION

We have developed here a new optical method for the determination of the waveform of vibration of an object in an arbitrary state of vibration. The method can in theory be used with objects which are transparent or opaque; diffusely or specularly reflecting. We have performed model experiments for a diffusely reflecting object vibrating at a low frequency. The low frequency was used to insure that the object followed the desired waveform, allowing us to compare the results directly with theory. As one increases the fundamental frequency of object vibration, two problems arise. First, the number of oscillation periods in the required exposure time increases, since the exposure time, for a given laser, geometrical setup and film type, will be independent of object frequency. This means that synchronism of the grating motion with the object motion is more critical. This, however, can still be achieved by electronic means. The second (and more serious) problem associated with higher frequencies is the number of grating lines required. The minimum number of grating lines one must have will be greater than the product of the fundamental object vibration frequency ($\frac{\omega}{2\pi}$) and the exposure time τ . (This is neglecting the width of the optical wave transmitted through the grating). For example, a vibration in the audio range, say 1000 Hz would require 5000 grating lines to allow for a 5 second exposure. A grating similar to the one used in our experiments (200 lines/inch) would have to be 25 inches long! This problem could be alleviated somewhat by using a shorter grating and making several passes. The relative alignment of the grating from pass to pass would be critical, however.

The extension of the method to waveforms with higher harmonic content involves improvement of the signal to noise ratio in the higher harmonics. One

definite improvement in this regard would be to utilize a diffraction grating with more ideal diffraction characteristics. Such gratings could, perhaps, be computer generated to produce the desired diffraction characteristics (see, for example, [43]). On the theoretical side, further work could be done on comparing the two waveform calculation methods presented in section III B.

Because of the stability requirements and overall complexity of the method developed here, the approach has admittedly a limited range of applicability. The method does, however, record a large amount of information in a single exposure. This fact, combined with the fundamental nature of the quantity measured, gives the method definite merit.

APPENDIX I (cf. p. 16)

A Probabilistic Method for Calculation of the Coefficients U_n

The coefficient $U_n(x)$ is basically the time average of the quantity $\exp[i\Phi - in\omega t]$. If we can view the argument of this exponential as a random function of time which is stationary and ergodic, then we may replace the time average (eq. 5) by an ensemble average:

$$U_n(x) = \int_{-\infty}^{\infty} e^{i\Phi - in\omega t} p(\Phi - n\omega t) d(\Phi - n\omega t) \quad (32)$$

This property, in connection with time-average holography, was first brought out by Kozma 6 , for the case $n = 0$.

As it stands, the variable in eq. 32 does not have stationary statistics. If, however, we consider our variable to be the periodic continuation of the phase difference $\Phi - n\omega t$ outside the interval $-\pi \leq \omega t < \pi$, then eq. 32 holds. In experimental terms, this is equivalent to periodically shifting the phase of the reference beam by an integer multiple of 2π , which clearly will not change the results. Notice that eq. 32 represents the Fourier transform of the probability distribution, evaluated at $v = \frac{1}{2\pi}$.

As an example, consider the wave form in figure 20(a). The periodic continuation of the phase difference $\Phi - n\omega t$ is shown in figure 20(b). From this it immediately follows that the probability distribution for $\Phi - n\omega t$ is two rectangle functions, as shown in figure 20(c). Finally, the desired function U_n is shown in figure 20(d) (the Fourier transform of 20 (c) evaluated at $v = \frac{1}{2\pi}$ and plotted as a function of vibration amplitude, a).

This method provides a means of quickly evaluating coefficients in simple common cases. It also provides some physical insight into the construction of the U_n 's. Notice finally, that the coefficient U_0 measures the characteristic function for the probability density function $p(\Phi)$, so that holography can be a useful tool for studying random statistical fluctuations, as well as deterministic process, as Kozma had recognized earlier [6].

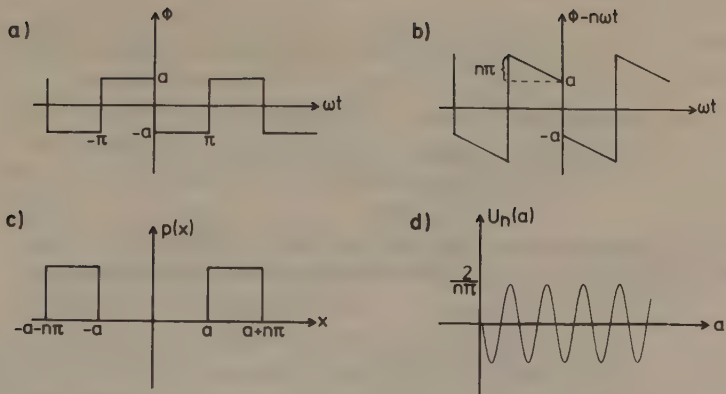


FIGURE 20: PROBABALISTIC DETERMINATION of the U_n

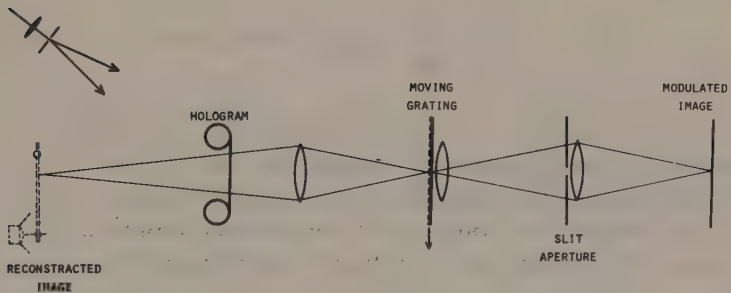


FIGURE 21: DISPLAY OF OBJECT MOTION PARAMETERS

APPENDIX 2 (cf. p. 16)

Expansion of the Coefficients U_n in terms of Bessel functions

From equations 14 and 16 it follows that

$$e^{iB \sin(\omega t + \alpha)} = \sum J_n(B) e^{in\alpha} e^{in\omega t} \quad (33)$$

This result can be easily generalized for an arbitrary Φ , if we first express Φ as a Fourier series:

$$\Phi = \sum_{m=0}^{\infty} B_m \sin(m\omega t + \alpha_m) \quad (34)$$

(This form of the Fourier series is more convenient in the present context.) From equations 33 and 34 it follows that

$$\begin{aligned} U &= e^{i\Phi} = \prod_{m=1}^{\infty} e^{iB_m \sin(m\omega t + \alpha_m)} \\ &= \prod_{m=1}^{\infty} \sum_{-\infty}^{\infty} J_n(B_m) e^{in\alpha_m} e^{inm\omega t} \end{aligned} \quad (35)$$

This expression is clearly very awkward to use to compute the coefficients U_n . However, we can "build" the coefficients U_n using the following approach:

We consider the addition of one Fourier component to an arbitrary waveform with known coefficients:

$$\begin{aligned} U &= e^{i[\Phi + B_m \sin(m\omega t + \alpha_m)]} \\ &= \left[\sum_n U_n e^{in\omega t} \right] \left[\sum_p J_p(B_m) e^{ip\alpha} e^{imp\omega t} \right] \\ &= \sum_n \left[\sum_p U_{n-mp} J_p(B_m) e^{ip\alpha} \right] e^{in\omega t} \end{aligned} \quad (36)$$

The quantity in brackets in the last line of eq. 36 is clearly the n^{th} Fourier coefficient of the wave given in the first line of eq. 36.

We can use eq. 36 to build, step by step, the coefficients corresponding to an arbitrary wave, from knowledge of the fourier transform (eq. 34). I. E., one uses eq. 36 with $\Phi \equiv 0$ (and hence $U_n = \delta_{n,0}$), and $m = 0$ to calculate a set of coefficients $\{U_n\}$. Then one uses eq. 36 with this new set of coefficients and $m = 1$ to generate another revised set of coefficients, and so on.

Eq. 36 can also be used to readily calculate the effects of changing one specific Fourier component (say, at a mechanical resonance frequency) in an arbitrary waveform, whose coefficients are already known (perhaps by another method).

APPENDIX 3 (cf. p. 20)

Optical Display of Object Motion Parameters

Consider the extended reconstruction setup shown in fig. 21. The reconstruction is imaged onto a grating moving with in general a different velocity, v' , from the velocity used during recording. After passage through the grating, the image amplitude will be given by (see eq. 25)

$$UG_2 = \sum_m \sum_n C_m D_n U_n(x) e^{i[2\pi v_o(n+m)x - \omega't]} \quad (37)$$

where $\omega' = 2\pi v_o v'$. This image is imaged into the output plane by a low-pass spatial filtering stage (the aperture in fig. 21) which suppresses all higher spatial frequencies in eq. 37, so that only the terms with $n+m = 0$ in eq. 37 survive.

Thus our reconstructed image amplitude, U_I , in the output plane is given as:

$$U_I(x, t) = \sum_n C_n D_{-n} U_n(x) e^{in\omega't} \quad (38)$$

If the coefficients $C_n D_{-n}$ in eq. 38 are essentially constant (independent of x and n), then the reconstructed image is a good replica of our original object $U(x, t)$, in both space and time. The possible shift in temporal frequency (when $\omega' \neq \omega$) allows reconstruction of $U(x, t)$ at a slower speed (or stationary).

If we consider varying these coefficients as a function of n , we can obtain other forms of information from the hologram reconstruction.

For example, we find from eq. 11 that

$$\left| \frac{\partial U}{\partial t} \right|^2 = \left| \frac{\partial \Phi}{\partial t} \right|^2$$

as one potentially useful output intensity.

If we express $U(x,t)$ in the Fourier representation (eq. 20), then we see that

$$\frac{\partial U}{\partial t} = \sum i n \omega U_n(x) e^{i n \omega t}$$

If we can arrange the coefficients in eq. 38 to vary according to:

$$C_n D_{-n} = n \omega \quad (39)$$

Then the intensity of reconstructed images as a function of time, will be proportional to the velocity squared of the object as a function of time. This quantity is interesting since it is proportional to the kinetic energy of the object. If, however, measurement of the velocity is desired, this choice of coefficients (eq. 39) leads to an ambiguity in the sign of the velocity. This could be eliminated by introducing a bias:

$$C_n D_{-n} = 1 - B n \omega$$

Where B is a small parameter. This leads to an output intensity given by

$$I = \left(1 - B \frac{\partial \Phi}{\partial t} \right)^2$$

This eliminates the sign ambiguity, provided B is small enough so that the quantity in the parenthesis never goes to zero.

The acceleration can be obtained by using coefficients given by

$$C_n D_{-n} = 1 - n^2 \omega^2 B$$

The resulting intensity would be

$$I = \left(1 - B \left[\frac{\partial \Phi}{\partial t} \right]^2 \right)^2 + B \left(\frac{\partial^2 \Phi}{\partial t^2} \right)^2$$

The first derivative must be measured first in order to evaluate the acceleration from the above expression.

These various changes in the diffraction coefficients could be implemented directly by using different gratings for reconstruction. Such gratings could be computer generated to produce the desired diffraction efficiencies [43]. A more flexible approach would be to add a spatial filter stage to the reconstruction setup. This method is described in reference [40].

APPENDIX 4 (cf. p. 35)

The Small Vibration Requirement

If we consider the form of our object wave U when the vibration amplitude is very small we see that:

$$U = e^{i\Phi} \cong 1 + i\Phi$$

In other words, for sufficiently small vibrations we expect the temporal Fourier coefficients of the object vibration, A_n , to be basically identical to the measured coefficients of the optical wave, U_n (except for the factor i and the additional 1 added to A_0). As the amplitude increases, the information spreads out into other orders.

As a numerical example, consider sinusoidal object motion:

$$\Phi = A \sin(\omega t)$$

If we perform a digital simulation, and calculate the coefficients U_n from eq. 16 we should be able to use the algorithm, eq. 19, to obtain back our original value of A . However, for $A = 2$ (corresponding to peak object motion of $\frac{\lambda}{2\pi}$, we find that measurement of all the U_n with $|n| \leq 3$ is necessary to measure A within 2 %. As A becomes even larger, the information spreads out even more. For $A = 10$ (peak object motion $\sim$ one wavelength), we need all the U_n with $3 \leq n \leq 13$ in order to measure A within 2 %.

It should be noted that one would usually deduce the amplitude information in the sinusoidal case from the location of the fringes in, say, the zeroth diffraction order (instead of using the algorithm, eq. 19). However, from the above argument it is clear that the effects of truncation strongly depend on the amplitude of vibration itself. Thus, when various Fourier components are present;

with in general different amplitudes, we can expect the truncation effect to upset the ratios of the components, which in turn changes the calculated waveform. For example, consider a case where higher frequencies are present.

The spreading of the information is even worse when higher frequencies are present. The worst case one might consider is a square wave object motion of amplitude $\frac{\lambda}{2}$ (peak-peak amplitude λ). Clearly such a motion is not detectable by optical means (at least if the motion is exactly a square wave) since the object will not appear to move at all. (Of course, real mechanical systems will always have a finite response time, so that a small optical effect will always be seen during the perhaps sudden (but not instantaneous) change in object position.)

To examine the severity of the above effect on a relatively smooth waveform, another digital simulation was worked through.

The waveform used was given by

$$\Phi = 4 \sin \omega t + \frac{4}{3} \sin 3 \omega t$$

(This is simply the first two terms of a series expansion of a square wave of amplitude π .)

The coefficients U_n were calculated, and the algorithm (eq. 19) was used to calculate Φ . If all the coefficients U_n were used, the algorithm converged to the correct waveform, as expected. If only the coefficients for $|n| \leq 5$ were measured, then the considerable error resulted, in both waveform and amplitude, as shown in fig. 22. In addition to the large distortion in the waveform itself, the calculated amplitude of vibration was off by about a factor of 4. These calculations did not consider the presence of noise or experimental error. In any real measurement of such a waveform, the results would probably be even worse.

Clearly, then, the vibrations measured by the multiplex method must be restricted to very small amplitudes, unless methods can be found to accurately measure the U_n for quite large values of n .

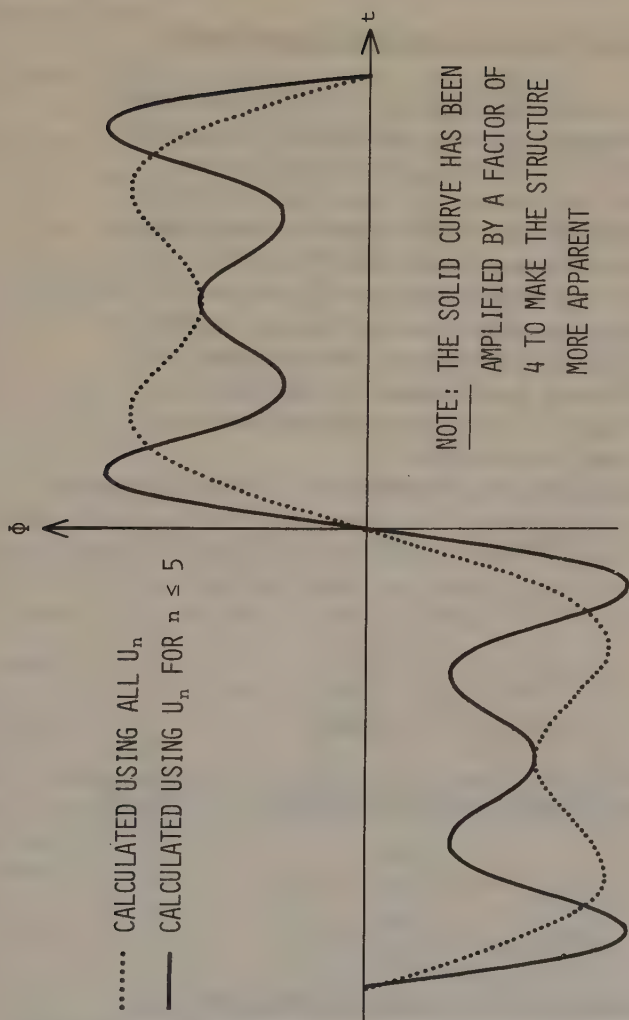


FIGURE 22: EFFECT OF TRUNCATION OF THE U_n

APPENDIX 5 (cf. p. 36)

Calculation of Phase from Intensity Measurements

There are two waves which will propagate through the entrance aperture of the photomultiplier; namely $C_n U_{nr}$ and $C_n U_n e^{i\beta_n}$, where C_n is the diffraction efficiency of the grating, U_{nr} is a reference wave reflected from the stationary object in the interferometer and diffracted into the n^{th} diffraction order by the (stationary) grating, U_n is the holographically reconstructed coefficient, and β_n is the phase difference between the two waves. (In this appendix, all coefficients are expressed in polar form; that is, U_n is the amplitude of the n^{th} coefficient and β_n the phase).

We measure four intensities:

$$I_1 = C_n^2 U_{nr}^2$$

$$I_2 = C_n^2 U_n^2$$

$$I_3 = C_n^2 \left| U_{nr} + U_n e^{i\beta_n} \right|^2 \quad (40)$$

$$I_4 = C_n^2 \left| U_{nr} e^{i\frac{\pi}{4}} + U_n e^{i\beta_n} \right|^2 \quad (41)$$

Straightforward expansion of equations 40 and 41 gives:

$$I_3 = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \beta_n \quad (42)$$

$$I_4 = I_1 + I_2 + 2\sqrt{I_1 I_2} \sin \beta_n \quad (43)$$

Equations 42 and 43 assume, however that the waveforms entering the aperture are perfectly correlated.

In actual practice, however, perfect correlation will not exist, for several reasons:

a) The hologram will not reproduce the object waveform exactly, due to grain-noise, non-linear effects, etc.

b) The object wave passing straight through the hologram plate will undergo some distortion.

c) General mechanical and atmospheric shifts can cause the phase difference to vary across the entrance aperture of the photomultiplier.

All these effects reduce the correlation (coherence) between the two waves, and so we should include a degree of coherence factor, γ , in equations 42 and 43:

$$I_3 = I_1 + I_2 + 2\sqrt{I_1 I_2} \gamma \cos \beta \quad (44)$$

$$I_4 = I_1 + I_2 + 2\sqrt{I_1 I_2} \gamma \sin \beta \quad (45)$$

Because the degree of coherence γ is generally unknown (and can vary from time to time because of turbulence and mechanical instabilities), neither eq. 44 or 45 can be used alone to calculate β_n . However, γ drops out if we consider the tangent:

$$\tan \beta = \frac{I_3 - I_1 - I_2}{I_4 - I_1 - I_2} \quad (46)$$

(The additive factor π ambiguity in the tangent function can be resolved by noting the sign of the numerator in eq. 46.)

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